

Tests for quantum contextuality in terms of q -entropies

Alexey E. Rastegin

Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia

The information-theoretic approach to Bell's theorem is developed with use of the conditional q -entropies. The q -entropic measures fulfill many properties similarly to the standard ones. In general, various formulations of Bell's theorem are based on the noncontextuality hypothesis. Using this hypothesis and properties of the conditional q -entropies, we derive information-theoretic inequalities of the Bell type for all $q \geq 1$. A recently proposed example of five observations for three-level system is explicitly analyzed within the context of q -entropic formulation. As is shown, quantum mechanics predicts the violation of an entire family of q -entropic inequalities of Bell's type.

Keywords: Bell theorem, Kochen–Specker theorem, contextuality hypothesis, conditional q -entropy, chain rule

I. INTRODUCTION

The concept of quantum entanglement is one of cornerstones of quantum theory. First of all, entangled states exhibit correlations that have no classical analogs. In effect, this fact was stressed by Schrödinger in his famous "cat paradox" paper [1]. Quantum nonlocality is brightly manifested in considering space-like separated measurements on different parts of an entangled quantum system. This celebrated issue in foundations of quantum mechanics has been initiated with the Einstein–Podolsky–Rosen paper [2]. The conventional version of the EPR argument has later been proposed by Bohm [3]. In the fruitful paper [4], Bell has made the next profound insight into the subject. Namely, no local hidden-variable theory can ever reproduce all of the predictions of quantum mechanics. This result allows to put the question in the category of experimentally tested statements [5]. For a discussion of basic experiments and future prospects, see [6] and references therein. For spin-1 quantum systems, the writers of Ref. [7] derived an inequality with five observations and called it the pentagram inequality.

It is common to demand that a hidden-variable model obey the so-called value definiteness and noncontextuality. Accepting the predictions of quantum mechanics, we have to renounce at least one of these requirements [8]. This result, which is usually referred to as the Kochen–Specker theorem, was independently derived by Bell [9] (for more details, see the review [10]). In a certain sense, the Bell and Kochen–Specker theorems are complemented to each other. As Leggett–Carg inequalities show [11], the predictions of quantum mechanics are also incompatible with the conjunction of the macroscopic realism and the uninvase measurability. Further, an incompatibility theorem has been established for a particular class of nonlocal hidden-variable theories [12]. Experimental results [13] have ruled out such hidden-variable theories. Some recent studies in reinterpretation of quantum mechanics are discussed in Refs. [23, 24]. Like the locality, the causality is also deeply rooted in our understanding of the macro world. The writers of Ref. [25] have recently argued the following. In quantum mechanics, it is possible to conceive situations in which a single event can be both, a cause and an effect of another one.

Due to impressive advances in both theory and experiment, quantum systems are now treated as tools for information processing [14]. Because of key role of entangled states, studies of quantum entanglement have today a technological significance. In Ref. [15], connections between violation of Bell's inequalities and security of quantum cryptography are summarized. In general, Bell's theorem can be expressed in many ways. Although the most known formulation deals with inequalities, the Greenberger–Horn–Zeilinger argument has provided a claim without inequalities [16]. Note that the EPR and GHZ states both give suitable tools in considering three-partite entanglement [17]. Using multilinear-contraction framework, Bell's inequalities can be naturally understood in geometric terms [18]. Information-theoretic formulations of Bell's theorem have found use as well [19]. Notions of information theory are indispensable tools in analyzing secure protocols [20]. Recently, entropic tests for quantum contextuality of three-level quantum systems are proposed in Ref. [21]. Entropic approach to Leggett–Carg inequalities has been considered as well [22].

As mentioned above, studies of Bell's theorem are important in own rights as well as in development of new technologies. The information-theoretical results are usually expressed in terms of the standard entropic functionals. Applying statistical methods in various fields, some extensions of the Shannon entropy were found to be useful. The Rényi [26] and Tsallis [27] entropies are both especially important generalizations. In the present paper, we derive information-theoretic formulations of Bell's theorem in terms of Tsallis conditional q -entropies for $q \geq 1$. The paper is organized as follows. In Sect. II, definitions and notation are introduced. We also prove one required property of the conditional q -entropy of order $q \geq 1$. Inequalities of Bell's type in terms of conditional q -entropies are derived in Sect. III. An important example with three-level quantum system is analyzed in Sect. IV. Within this example, the violation of the obtained inequalities could be tested in the experiment. In Sect. V, we conclude the paper with a summary of results.

II. TSALLIS ENTROPIES AND CONDITIONAL q -ENTROPIES

In this section, we briefly recall definitions of the Tsallis entropies and corresponding conditional entropies. Required properties of these entropic functionals are discussed as well. Let the variable A take values on the set Ω_A with corresponding probability distribution $\{p(a) : a \in \Omega_A\}$. The Tsallis entropy of degree $q > 0 \neq 1$ is defined by [27]

$$H_q(A) := \frac{1}{1-q} \left(\sum_{a \in \Omega_A} p(a)^q - 1 \right). \quad (1)$$

With the factor $(2^{1-q} - 1)^{-1}$ instead of $(1-q)^{-1}$, this entropic form was derived from several axioms by Havrda and Charvát [28]. It is convenient to rewrite the entropy (1) as

$$H_q(A) = - \sum_{a \in \Omega_A} p(a)^q \ln_q p(a) = \sum_{a \in \Omega_A} p(a) \ln_q \frac{1}{p(a)}. \quad (2)$$

Here the q -logarithm $\ln_q x = (x^{1-q} - 1)/(1-q)$ is defined for $q > 0 \neq 1$, $x > 0$, and obeys $\ln_q(1/x) = -x^{q-1} \ln_q x$. In the limit $q \rightarrow 1$, we obtain $\ln_q x \rightarrow \ln x$ and the Shannon entropy

$$H_1(A) = - \sum_{a \in \Omega_A} p(a) \ln p(a). \quad (3)$$

For brevity, we will usually omit the symbol of the set Ω_A in entropic sums. Properties of quantum counterpart of the entropy (1) are examined in Ref. [29] and, with a broader perspective, in the book [30].

Let B be another variable taking values on the set Ω_B with probability distribution $\{p(b) : b \in \Omega_B\}$. The entropy of A conditional on knowing B is defined as [31]

$$H_1(A|B) := \sum_b p(b) H_1(A|b) = - \sum_a \sum_b p(a, b) \ln p(a|b), \quad (4)$$

where $H_1(A|b) := - \sum_a p(a|b) \ln p(a|b)$ and $p(a|b) = p(a, b) p(b)^{-1}$ according to the Bayes rule. The quantity (4) will be referred to as the standard conditional entropy. By means of the particular functional

$$H_q(A|b) := - \sum_a p(a|b)^q \ln_q p(a|b) = \sum_a p(a|b) \ln_q \frac{1}{p(a|b)}, \quad (5)$$

we define the conditional q -entropy [32, 33]

$$H_q(A|B) := \sum_b p(b)^q H_q(A|b). \quad (6)$$

In the limit $q \rightarrow 1$, this definition is reduced to Eq. (4). Below, we will extensively use the following properties of the entropic function (6). For all $q > 0$, one satisfies

$$H_q(A, B) = H_q(B|A) + H_q(A) = H_q(A|B) + H_q(B). \quad (7)$$

This formula expresses the chain rule for the conditional q -entropy [32]. It can easily be derived in line with the definitions (2) and (6) by means of the identity

$$\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y. \quad (8)$$

Quantum violation of the Clauser–Horne–Shimony–Holt inequality is limited from above by the Tsirel'son bound [34]. This bound can be derived from the assumption that the chain rule holds for a generalized mutual information introduced in Ref. [35].

The chain rule (7) can further be extended to more than two variables (see theorem 2.4 in Ref. [32]):

$$H_q(A_1, A_2, \dots, A_n) = \sum_{j=1}^n H_q(A_j | A_{j-1}, \dots, A_1). \quad (9)$$

Using Eq. (7) and non-negativity of the conditional q -entropy, we immediately obtain

$$H_q(A) \leq H_q(A, B), \quad H_q(B) \leq H_q(A, B). \quad (10)$$

In the next section, we will also use inequalities of the following form.

Lemma 1 For real $q \geq 1$ and integer $n \geq 1$, the conditional q -entropy satisfies

$$H_q(A|B_1, \dots, B_{n-1}, B_n) \leq H_q(A|B_1, \dots, B_{n-1}) . \quad (11)$$

Proof. First, we prove the claim for $n = 2$. The conditional q -entropy $H_q(A|B, C)$ can be rewritten as

$$H_q(A|B, C) = \sum_{ab} p(b)^q \sum_c \left(\frac{p(b, c)}{p(b)} \right)^q f_q(p(a|b, c)) , \quad (12)$$

where the function $f_q(x) := (x^q - x)/(1 - q)$ is concave. As the probabilities $p(c|b) = p(b, c)/p(b)$ are summarized to 1, we get $p(c|b)^q \leq p(c|b)$ for $q \geq 1$. So, the sum with respect to c obeys

$$\sum_c p(c|b)^q f_q(p(a|b, c)) \leq \sum_c p(c|b) f_q(p(a|b, c)) \leq f_q \left(\sum_c p(c|b) p(a|b, c) \right) , \quad (13)$$

in line with Jensen's inequality. Since the numbers $p(c|b) p(a|b, c) = p(b, c) p(b)^{-1} p(a, b, c) p(b, c)^{-1} = p(a, b, c) p(b)^{-1}$ are summarized to $p(a, b) p(b)^{-1} = p(a|b)$, the right-hand side of Eq. (13) reads $f_q(p(a|b))$. Combining this with Eq. (12) then gives

$$H_q(A|B, C) \leq \sum_{ab} p(b)^q f_q(p(a|b)) = H_q(A|B) . \quad (14)$$

By a parallel argument, we easily have the case $n = 1$, namely

$$H_q(A|B) \leq H_q(A) . \quad (15)$$

The proof of Eq. (11) is completed by obvious extension with respect to n . ■

Note that there exists another form of the conditional q -entropy [32]. However, this form does not succeed some useful relations including the chain rule. Further properties of both forms of the conditional q -entropy are discussed in the papers [32, 33]. In the following, the conditional q -entropy of order $q \geq 1$ is used for expressing inequalities of Bell's type.

III. ENTROPIC INEQUALITIES OF THE BELL TYPE

In this section, we formulate Bell's theorem in terms of the conditional q -entropies. By \mathcal{A} and \mathcal{B} , we mean two widely separated systems. We assign measurable quantities A, A' to the system \mathcal{A} and measurable quantities B, B' to the system \mathcal{B} . Values of these quantities are respectively denoted as a, a' and b, b' . The noncontextuality hypothesis implies that, in each run of the experiment, all four quantities have definite values independently of observation. Only two values, one from \mathcal{A} and one from \mathcal{B} , are determined in each run. Nevertheless, there exist a joint probability distribution $p(a, b', a', b)$, from which we have relations of the form

$$p(a, b) = \sum_{b', a'} p(a, b', a', b) . \quad (16)$$

As the systems \mathcal{A} and \mathcal{B} are widely separated, a measurement on one should not disturb the other. Such no-disturbance assumption is obviously based on locality.

Using inequalities of the form (10) and reexpressing $H_q(A, B', A', B)$ with respect to the chain rule (9), we have

$$H_q(A, B) \leq H_q(A, B', A', B) = H_q(A|B', A', B) + H_q(B'|A', B) + H_q(A'|B) + H_q(B) . \quad (17)$$

Subtracting $H_q(B)$ and using Eq. (7), one further obtains

$$H_q(A|B) \leq H_q(A|B', A', B) + H_q(B'|A', B) + H_q(A'|B) . \quad (18)$$

According to Lemma 1, for $q \geq 1$ we write

$$H_q(A|B', A', B) \leq H_q(A|B') , \quad H_q(B'|A', B) \leq H_q(B'|A') . \quad (19)$$

Combining these relations with Eq. (18), we have arrived at entropic Bell's inequality

$$H_q(A|B) \leq H_q(A|B') + H_q(B'|A') + H_q(A'|B) , \quad (20)$$

which holds for $q \geq 1$. In the case $q = 1$, the formula (20) becomes entropic Bell's inequality derived by Braunstein and Caves [19]. So, we have obtained a one-parametric extension of their entropic relation in terms of the conditional q -entropies. The Bell states indispensable in quantum information are simplest example of two separated qubits. Note that entangled states of such a kind are arisen in study of cloning process [36]. Like Bell's theorem, the no-cloning theorem [36, 37] is one of conceptual no-go statements in quantum theory. No-cloning results have been formulated for mixed states [39], for the case of additional information in ancilla [38], and for measurement statistics [40]. A role of approximate cloning in quantum cryptography is reviewed in Ref. [41].

The above reasons with two spacelike separated systems are closely related to initial Bell's formulation [4, 9]. A more general approach has been examined in Refs. [7, 21]. It was shown with five observations for three-level quantum system, in both average-value statement [7] and entropic one [21]. In principle, any massive spin-1 particle may provide a physical example of such a kind. The discussed approach, posed formally, is this. Imagine a physical system, on which we could perform measurements chosen from the set $\{A_1, A_2, \dots, A_n\}$. Possible outcomes of j -th measurement are denoted by label a_j . The noncontextuality hypothesis implies that there exists a joint probability distribution for the outcomes of all observations, namely $p(a_1, a_2, \dots, a_n)$. The latter should recover all the measurable probabilities as its marginals. So, for any jointly measurable subset $\{A_k : k \in L\}$ with the corresponding set of indices $L = \{k_1, k_2, \dots, k_m\}$, we then have

$$p(a_{k_1}, a_{k_2}, \dots, a_{k_m}) = \sum_{a_j: j \notin L} p(a_1, a_2, \dots, a_n) . \quad (21)$$

Here the summation is taken over the outcomes of all those measurements A_j that are not in the jointly measurable subset $\{A_k : k \in L\}$. The inequality (20) is extended in the following way.

Theorem 1 *Let the measurable probabilities for observations A_1, A_2, \dots, A_n be all recovered as marginals of the joint probability distribution. Let subset $\{A_k : k \in L\}$ of cardinality m be such that each pair $\{A_{k_{i-1}}, A_{k_i}\}$ is jointly measurable. For $q \geq 1$, there holds*

$$H_q(A_{k_1}|A_{k_m}) \leq \sum_{i=2}^m H_q(A_{k_{i-1}}|A_{k_i}) . \quad (22)$$

This statement can be derived by means of obvious extension of the reasons from Eqs. (17)–(20). We refrain from presenting the details here. The writers of Ref. [21] have also considered an important case of five measurable observations for three-level quantum system which we called *qutrit*. In terms of the conditional q -entropies of order $q \geq 1$, we have

$$H_q(A_1|A_5) \leq H_q(A_1|A_2) + H_q(A_2|A_3) + H_q(A_3|A_4) + H_q(A_4|A_5) . \quad (23)$$

The basic inequality of Ref. [21] is obtained from Eq. (23) for $q = 1$. Thus, we have obtained an extension of the main relation of Ref. [21] to the case of conditional q -entropies of order $q \geq 1$. To compare Eq. (22) with predictions of quantum mechanics, we put the quantity

$$\mathcal{C}_q := H_q(A_{k_1}|A_{k_m}) - \sum_{i=2}^m H_q(A_{k_{i-1}}|A_{k_i}) . \quad (24)$$

For $q = 1$, this quantity was introduced in Ref. [21]. The inequality (22) is then rewritten as $\mathcal{C}_q \leq 0$. If predictions of quantum mechanics do sometimes lead to strictly positive \mathcal{C}_q , then the noncontextuality hypothesis fails. In such a case, the quantity \mathcal{C}_q characterizes an amount of violation of the inequality (22). As was argued in Ref. [21], violation of the inequality (22) implies violation of the corresponding pentagram inequality of Ref. [7], but the converse is not true. Further, obtained findings could be verified in the experiment. In the next section, we consider an example of five observations for qutrit.

IV. VIOLATION OF ENTROPIC INEQUALITIES WITH FIVE QUTRIT OBSERVATIONS

Following Refs. [7, 21], we consider five projectors of the form $|A_k\rangle\langle A_k|$ with the normalized eigenvectors

$$|A_1\rangle = \frac{1}{\sqrt{2}\cos\alpha} \begin{pmatrix} \sqrt{\cos 2\alpha} \\ \sin\alpha \\ \cos\alpha \end{pmatrix}, \quad |A_2\rangle = \begin{pmatrix} 0 \\ \cos\alpha \\ -\sin\alpha \end{pmatrix}, \quad |A_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (25)$$

$$|A_4\rangle = \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad |A_5\rangle = \frac{1}{\sqrt{2}\cos\alpha} \begin{pmatrix} \sqrt{\cos 2\alpha} \\ \sin\alpha \\ -\cos\alpha \end{pmatrix}, \quad (26)$$

TABLE I: The maximal values of \mathcal{C}_q and \mathcal{R}_q for several q .

q	1.0	1.1	1.2	1.4	1.6	1.8	2.0	2.5	3.0	5.0	8.0	11.0
$\max \mathcal{C}_q$	0.0631	0.0779	0.0898	0.1049	0.1111	0.1113	0.1079	0.0924	0.0759	0.0383	0.0212	0.0146
$\max \mathcal{R}_q$	0.0911	0.1164	0.1387	0.1733	0.1960	0.2093	0.2157	0.2143	0.2024	0.1632	0.1494	0.1462
α_{\max}	0.1698	0.1802	0.1880	0.1987	0.2051	0.2085	0.2099	0.2067	0.1982	0.1557	0.1205	0.1017
θ_{\max}	0.2366	0.2684	0.2943	0.3327	0.3585	0.3761	0.3880	0.4014	0.3996	0.3345	0.2639	0.2247

where $\alpha \in (0; \pi/4)$. The five vectors satisfy orthogonality constraints

$$\langle A_1|A_2\rangle = \langle A_2|A_3\rangle = \langle A_3|A_4\rangle = \langle A_4|A_5\rangle = \langle A_5|A_1\rangle = 0 . \quad (27)$$

Hence, each pair of the form $\{|A_{k-1}\rangle\langle A_{k-1}|, |A_k\rangle\langle A_k|\}$ is jointly measurable. Eigenvalues 1 and 0 of the projector $|A_k\rangle\langle A_k|$ respectively correspond to outcomes "yes" and "no", when measured quantum state passes the test of being the state $|A_k\rangle$. The vectors (25)–(26) also obey

$$\langle A_1|A_4\rangle = \langle A_5|A_2\rangle , \quad \langle A_1|A_3\rangle = \langle A_5|A_3\rangle . \quad (28)$$

Further, we choose the measured state

$$|\psi\rangle = \begin{pmatrix} \sin \theta \\ \cos \theta \\ 0 \end{pmatrix} , \quad (29)$$

for which $\langle A_1|\psi\rangle = \langle A_5|\psi\rangle$ and $\langle A_2|\psi\rangle = \langle A_4|\psi\rangle$. Some intuitive reasons for such a configuration are briefly discussed in Ref. [21].

Let $|\psi\rangle$ be the state right before measurement. The observation A_k leads to the outcomes $a_k = 1$ and $a_k = 0$ with probabilities $|\langle A_k|\psi\rangle|^2$ and $1 - |\langle A_k|\psi\rangle|^2$, respectively. According to the projection postulate, the normalized state right after measurement should be $|A_k\rangle$ for $a_k = 1$ and

$$(1 - |\langle A_k|\psi\rangle|^2)^{-1/2} \left\{ |\psi\rangle - |A_k\rangle\langle A_k|\psi\rangle \right\} \quad (30)$$

for $a_k = 0$. Hence the context for next observations is determined. If the next observation is A_j , we calculate the conditional probabilities and, further, the corresponding entropy $H_q(A_j|A_k)$. In this way, one estimates the characteristic quantity

$$\mathcal{C}_q = H_q(A_1|A_5) - H_q(A_1|A_2) - H_q(A_2|A_3) - H_q(A_3|A_4) - H_q(A_4|A_5) . \quad (31)$$

The main result is that the inequality (23) is violated for certain values of the parameters α and θ . We do not solve analytically the problem of finding a joint parametric domain in which $\mathcal{C}_q > 0$. For given parameters, however, the quantity \mathcal{C}_q is easy to numerical estimation. Some numerical results are summarized below. Here, one should measure positive values of \mathcal{C}_q with a natural scale of entropic values. To do so, we will relate \mathcal{C}_q with the number $\ln_q 2$, which represents the maximal binary q -entropy. Thus, the results are reported in terms of the relative quantity

$$\mathcal{R}_q := (\ln_q 2)^{-1} \mathcal{C}_q . \quad (32)$$

In Table I, the maximal values of \mathcal{C}_q and \mathcal{R}_q are shown for several values of the parameter q . The values α_{\max} and θ_{\max} , which correspond to the maximal violation, are given as well. In relative entropic size, the maximal violation of Eq. (23) is sufficiently large for all the presented values of q . The standard case $q = 1$ has previously been reported in Ref. [21]. For convenience of comparing with values $q > 1$, we insert this case in the table as well. As is seen in Table I, the values α_{\max} and θ_{\max} are dependent on q . In given experimental setting, a fixed value of α and few values of θ would be rather available. On Fig. 1, a dependence of \mathcal{R}_q on θ is presented for $\alpha = 0.1885$ and five values of the parameter q . It shows that violation of Eq. (23) is significant for many values $q \geq 1$. In the same experimental setup, therefore, one could test violation of an entire family of q -entropic inequalities of Bell's type. Thus, the obtained results can be regarded as an extension and development of theoretical findings of Refs. [7, 21].

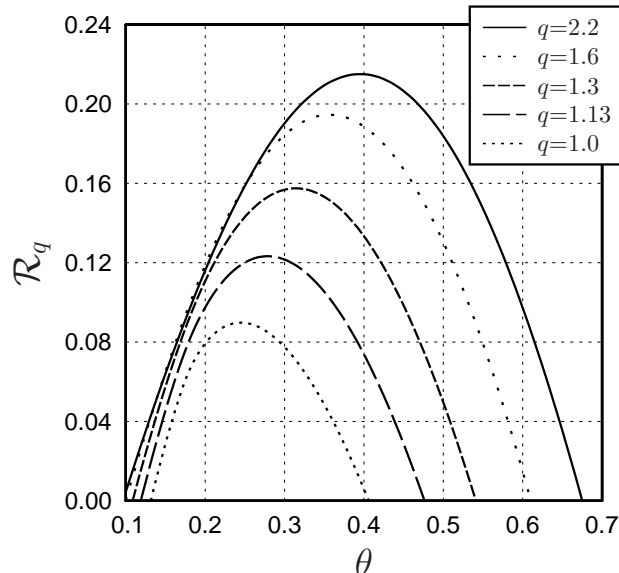


FIG. 1: The relative quantity \mathcal{R}_q versus θ for $\alpha = 0.1885$ and five values of q , namely $q = 1.0; 1.13; 1.3; 1.6; 2.2$. For each case, only positive values of \mathcal{R}_q are shown.

V. CONCLUSIONS

In the paper, we have expressed Bell's theorem in terms of the conditional q -entropies of order $q \geq 1$. Formally, the presented inequalities are based on two useful properties of the conditional q -entropy. One of them is the well-known chain rule. Another property is proved as Lemma 1. This result holds for $q \geq 1$ and generalizes analogous property of the standard conditional entropy. From the physical viewpoint, the noncontextuality hypothesis is a key ingredient of the derivation. Assuming existence of a joint probability distribution for the outcomes of all observations, we have arrived at a principal conclusion. Namely, the corresponding conditional q -entropies of order $q \geq 1$ should satisfy inequalities of the form (22). This claim generalizes the previous entropic formulations of Bell's theorem. In particular, the inequality (20) is a q -parametric extension of the main result of Ref. [19]. Within the context of conditional q -entropies, we have discussed an example of five observations for qutrit. With the standard conditional entropy, this example was examined in Ref. [21]. Adding to the results of Ref. [21], we have shown that the noncontextuality hypothesis leads to an entire family of q -entropic inequalities of Bell's type. It turns out that these inequalities are incompatible with the predictions of quantum mechanics for many values of the parameters. The obtained conclusions for various values $q \geq 1$ could be tested in the experiments. For conventional Bell's inequality in terms of average values, quantum violation is limited by the Tsirel'son bound. It would be interesting to obtain upper bounds on a possible violation of q -entropic inequalities of Bell's type. Due to the role of entangled states in quantum information processing, results of such a kind may also have a practical significance.

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